Diffusion of Radiation in a Semi-Infinite Medium in the Case of Anisotropic Scattering. I.

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In 1951 Sobolev[1] proposed a new method of solving various problems of the theory of radiative transfer based on the probability of emission of a quantum from the medium. Sobolev applied this method to the solution of a series of special problems of the theory of radiative transfer and developed a general statistical interpretation for the problem of radiative transfer.[2] In particular, he solved completely the problem of radiation diffusion in a plane layer for any radiation sources, whose power is dependent only on the depth, for isotropic scattering.[3] The present paper investigates the problem of radiation diffusion in a semi-infinite medium by the probability method in the case of anisotropic scattering.

In astrophysics the case of anisotropic scattering is encountered during investigations of the optical properties of planetary atmospheres and dust nebulae. Investigations of stellar atmospheres indicate that such a case exists for the scattering of light on free electrons and also at spectrum line frequencies.

1. The Probability of the Emission of a Quantum from the Medium

Let us now introduce quantity $p(\tau, \eta', \eta, \varphi' - \varphi)d\omega$—the probability that a quantum absorbed at an optical depth τ from the direction forming an angle arc $\cos\eta'$ with the outer normal to the layers and having azimuth φ', is emitted from the medium through boundary $\tau = 0$ under angle arc $\cos\eta$ to the normal and with the azimuth φ within solid angle $d\omega$. Together with this probability, we also introduce $q(\tau, \eta', \eta, \varphi' - \varphi)d\omega$—the probability that a quantum, emitted to the optical depth τ in the direction which forms the angle arc $\cos\eta'$ with the outer normal and has the azimuth φ', leaves the medium at an angle arc $\cos\eta$ to the normal and with the azimuth φ within the solid angle $d\omega$.

If, in the sense of the probability, the scattering indicatrix $x(\gamma)$ and the ratio of the coefficient of scattering to the extinction coefficient λ are introduced, the following relationship is obtained:

$$p(\tau, \eta', \eta, \varphi' - \varphi) = \frac{\lambda}{4\pi}\int_0^{2\pi} d\varphi'' \int_{-1}^{+1} x(\gamma)\, q(\tau, \eta'', \eta, \varphi'' - \varphi)d\eta'' \quad (1)$$

where

$$\cos\gamma = \eta''\eta' + \sqrt{(1-\eta''^2)(1-\eta'^2)}\cos(\varphi'' - \varphi') \quad (2)$$

The expression for the function $q(\tau, \eta', \eta, \varphi' - \varphi)$ can be derived readily from $p(\tau, \eta', \eta, \varphi' - \varphi)$ on the basis of the fact that this quantity

$$e^{-\frac{\tau-\tau'}{\eta'}}\frac{d\tau'}{\eta'} \qquad \text{for } \eta' > 0$$

$$-e^{-\frac{\tau-\tau'}{\eta'}}\frac{d\tau'}{\eta'} \qquad \text{for } \eta' < 0 \quad (3)$$

determines the probability that the quantum, emitted at depth τ in the direction forming an angle arc $\cos\eta'$ with the normal, is absorbed in the layer from τ' to $\tau' + d\tau'$. Namely

$$q(\tau, \eta', \eta, \varphi' - \varphi) = \int_0^\tau e^{-\frac{\tau-\tau'}{\eta'}} p(\tau', \eta', \eta, \varphi' - \varphi)\frac{d\tau'}{\eta'} + \delta(\eta' - \eta)\cdot\delta(\varphi' - \varphi)\cdot e^{-(\tau/\eta)} \qquad (\eta' > 0) \quad (4)$$

$$q(\tau, \eta', \eta, \varphi' - \varphi) = -\int_\tau^\infty e^{-\frac{\tau-\tau'}{\eta'}} p \times (\tau', \eta', \eta, \varphi' - \varphi)\frac{d\tau'}{\eta'} \qquad (\eta' < 0) \quad (5)$$

In (4) δ is the Dirac function and quantity $e^{-(\tau/\eta)}$ determines the emission probability of the quantum from depth τ emitted without scattering at this depth at an angle arc $\cos\eta$ to the normal. Equations (4) and (5) give

$$\eta'\frac{\partial q(\tau, \eta', \eta, \varphi' - \varphi)}{\partial\tau} = -q(\tau, \eta', \eta, \varphi' - \varphi) + p(\tau, \eta', \eta, \varphi' - \varphi) \quad (6)$$

The existence of (1), (4), and (5) indicates that it is sufficient to study only one of the functions $p(\tau, \eta', \eta, \varphi' - \varphi)$ and $q(\tau, \eta', \eta, \varphi' - \varphi)$, since the other is determined from the given representations. Function $p(\tau, \eta', \eta, \varphi' - \varphi)$, though, is investigated in the following.

2. Equations for $p(\tau, \eta', \eta, \varphi' - \varphi)$

Substitution of (4) and (5) in (1) gives

$$p(\tau, \eta', \eta, \varphi' - \varphi) = \frac{\lambda}{4\pi}\int_0^{2\pi} d\varphi''\left[\int_0^1 x(\gamma)d\eta''\int_0^\tau p(\tau', \eta'', \eta, \varphi'' - \varphi)\times e^{-\frac{\tau-\tau'}{\eta''}}\frac{d\tau'}{\eta''} - \int_{-1}^0 x(\gamma)d\eta''\int_\tau^\infty p(\tau', \eta'', \eta, \varphi'' - \varphi)\times e^{-\frac{\tau-\tau'}{\eta''}}\frac{d\tau'}{\eta''}\right] + \frac{\lambda}{4\pi}x(\gamma_1)e^{-(\tau/\eta)} \quad (7)$$

where

$$\cos\gamma_1 = \eta\eta' + \sqrt{(1-\eta^2)(1-\eta'^2)}\cos(\varphi' - \varphi) \quad (8)$$

Equation (7) can be derived readily on the basis of the following:

$$p(\tau, \eta', \eta, \varphi' - \varphi) = \frac{B(\tau, -\eta', \eta, \varphi' - \varphi)}{\pi S} \quad (9)$$

which was found by Sobolev.[1] In (9), $B(\tau, -\eta', \eta, \varphi' - \varphi)$ is the source function in the problem of the luminosity of the medium illuminated by parallel rays; πS is the magnitude of flux of incident radiation per unit area, perpendicular to the rays and located at the boundary of medium $\tau = 0$. It is pointed out also that the following equation is valid:

$$q(\tau, \eta', \eta, \varphi' - \varphi) = \frac{I(\tau, -\eta', \eta, \varphi' - \varphi)}{\pi S} \quad (10)$$

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where $I(\tau, -\eta', \eta, \varphi' - \varphi)$ is the intensity of radiation in the problem of the luminosity of the medium illuminated by parallel rays.

Let us now derive another equation for the function $p(\tau, \eta', \eta, \varphi' - \varphi)$ using its probabilistic meaning. Let us first determine the probability of a quantum emission from the optical depth $\tau + \Delta\tau$ assuming that the probability of emission from depth τ is known. In such a case it is assumed that the quantum is emitted from optical depth τ and that it then passes through an additional layer of thickness $\Delta\tau$. Hence

$$p(\tau + \Delta\tau, \eta', \eta, \varphi' - \varphi) = p(\tau, \eta', \eta, \varphi' - \varphi) \times \left(1 - \frac{\Delta\tau}{\eta}\right) + \int_0^1 \int_0^{2\pi} p(\tau, \eta', \eta'', \varphi' - \varphi'') \times p(0, \eta'', \eta, \varphi'' - \varphi) \frac{\Delta\tau}{\eta''} d\eta'' d\varphi'' \quad (11)$$

where the first term of the right side takes into account the quanta passing through the additional layer without absorption and the second term takes into account the quanta absorbed in the additional layer which are re-emitted from the medium after scattering in the given direction. Equation (11) gives for $\Delta\tau \rightarrow 0$:

$$\frac{\partial p(\tau, \eta', \eta, \varphi' - \varphi)}{\partial\tau} = -\frac{1}{\eta} p(\tau, \eta', \eta, \varphi' - \varphi) + \int_0^1 \int_0^{2\pi} p(\tau, \eta', \eta'', \varphi' - \varphi'') \cdot p(0, \eta'', \eta, \varphi'' - \varphi) \frac{d\tau''}{\eta''} d\varphi'' \quad (12)$$

The following relationship must be added to the functional equation (12):

$$p(0, \eta', \eta, \varphi' - \varphi) = \frac{\lambda}{4\pi} \times \left[x(\gamma_1) + \frac{\eta}{\pi} \int_0^{2\pi} d\varphi'' \int_0^1 x(\gamma)\rho(\eta'', \eta, \varphi'' - \varphi) d\eta''\right] \quad (13)$$

which is derived readily from (7) for $\tau = 0$. It is noted that here

$$\cos\gamma = -\eta'\eta'' + \sqrt{(1 - \eta'^2)(1 - \eta''^2)} \times \cos(\varphi' - \varphi'') \qquad \eta'' > 0$$

and $\rho(\eta', \eta, \varphi' - \varphi)$ is the coefficient of diffuse reflection for the given medium. Equation (13) can be derived also from physical representations, considering that $p(0, \eta', \eta, \varphi' - \varphi)$ is combined of the probability of the quantum emission directly from the boundary layer and the probability of scattering of the quantum by this layer toward the medium with the subsequent diffuse reflection of the quantum by the medium. Equation (13) is derived by employing the following equation:

$$\rho(\eta'', \eta, \varphi'' - \varphi) \cdot \eta'' = \pi \cdot \int_0^\infty p(\tau, -\eta'', \eta, \varphi'' - \varphi) e^{-(\tau/\eta'')} \frac{d\tau}{\eta} \quad (14)$$

which is derived readily taking into account the probabilistic meaning of the quantities contained.

3. Luminosity of the Medium Illuminated by Parallel Rays

Let the medium be illuminated by a flux of parallel rays incident on the boundary of the medium at an angle arc cosζ to the normal and with the azimuth φ_0. These rays produce a luminosity equal to πS of the plane perpendicular to them. Hence, taking into account that the quantity

$$\pi S e^{-(\tau/\zeta)} d\tau \quad (15)$$

determines the amount of energy absorbed by elementary volume having a cross section of 1 cm² and optical thickness $d\tau$ for an optical depth τ per second from the direction forming an angle arc cos $(-\zeta)$ with the normal and having the azimuth φ_0, we have

$$I(0, \eta, \zeta, \varphi - \varphi_0) = \pi S \cdot \int_0^\infty e^{-(\tau/\zeta)} p(\tau, -\zeta, \eta, \varphi_0 - \varphi) \frac{d\tau}{\eta} \quad (16)$$

where $I(0, \eta, \zeta, \varphi - \varphi_0)$ is the intensity of radiation emitted from the medium at an angle arc cosη to the normal and with azimuth φ. On the other hand, as already known, it is possible to write

$$I(0, \zeta, \eta, \varphi_0 - \varphi) = \int_0^\infty e^{-(\tau/\zeta)} B(\tau, \zeta, \eta, \varphi_0 - \varphi) \frac{d\tau}{\zeta} \quad (17)$$

Using (9) and comparing (16) and (17) we have

$$I(0, \eta, \zeta, \varphi - \varphi_0) \cdot \eta = I(0, \zeta, \eta, \varphi_0 - \varphi) \cdot \zeta \quad (18)$$

From the definition of the coefficient of diffuse reflection, in agreement with the relation

$$I(0, \eta, \zeta, \varphi - \varphi_0) = S\rho(\eta, \zeta, \varphi - \varphi_0)\zeta \quad (19)$$

we have finally

$$\rho(\eta, \zeta, \varphi - \varphi_0) = \rho(\zeta, \eta, \varphi_0 - \varphi) \quad (20)$$

Equation (20) shows that the coefficient of diffuse reflection is a symmetrical function of the angles of incidence and reflection. This property expresses the principle of reciprocity for this case. The validity of Eq. (20) in the case of a spherical indicatrix of scattering has been proved in the past by Sobolev.[2]

It should be pointed out that so far we investigate the problem of radiation diffusion in a semi-infinite medium under the assumption that quantities λ and $x(\gamma)$ are independent of the optical depth τ. However, if this assumption is neglected and the investigation follows lines analogous to the one set forth, then Eq. (20) is obtained again. Hence, the principle of reciprocity in terms of (20) is valid in a semi-infinite medium for any arbitrary scattering indicatrix and for any arbitrary variation of the optical characteristics with the depth.

Let us assume now that the scattering indicatrix is expanded in Legendre's polynomials

$$x(\gamma) = \sum_{i=0}^{n} x_i P_i(\cos\gamma) \quad (21)$$

Using the addition theorem for Legendre's polynomials and taking (2) into account we have

$$x(\gamma) = \sum_{m=0}^{n} (2 - \delta_{0,m}) \cos m(\varphi'' - \varphi) \times \sum_{i=m}^{n} c_i^m P_i^m(\eta'') P_i^m(\eta) \quad (22)$$

where

$$\delta_{0,m} = 1 \quad \text{for} \quad m = 0$$

$$\delta_{0,m} = 0 \quad \text{for} \quad m > 0$$

$$c_i^m = x_i \frac{(i - m)!}{(i + m)!}$$

$$(i = m, m + 1, \ldots, \quad \text{and} \quad 0 \leqslant m \leqslant n)$$

where $P_i^m(\eta)$ is the associated Legendre function.

The magnitude of the coefficient of diffuse reflection is found through (12) assuming that $\eta' = -\zeta$, where $\zeta > 0$

and $\varphi' = \varphi_0$. Multiplication of this equation by $e^{-(\tau/\zeta)}d\tau$ and integration over τ within the limits of 0 to ∞ give

$$(\eta + \zeta)\rho(\eta, \zeta, \varphi - \varphi_0) = \pi\rho(0, -\zeta, \eta, \varphi_0 - \varphi) + \zeta \int_0^1 \int_0^{2\pi} \rho(\eta'', \zeta, \varphi_0 - \varphi'') \times p(0, \eta'', \eta, \varphi'' - \varphi)d\eta''d\varphi'' \quad (23)$$

Equation (12) implies that function $p(\tau, \eta', \eta, \varphi' - \varphi)$ for the scattering indicatrix of form (21) can be expressed in the following terms:

$$p(\tau, \eta', \eta, \varphi' - \varphi) = \sum_{m=0}^{n} p_m(\tau, \eta', \eta) \cos m(\varphi' - \varphi) \quad (24)$$

where quantities $p_m(\tau, \eta', \eta)$ satisfy the following equations:

$$\frac{\partial p_m(\tau, \eta', \eta)}{\partial \tau} = -\frac{1}{\eta} p_m(\tau, \eta', \eta) + \pi(1 + \delta_{0,m}) \times \int_0^1 p_m(\tau, \eta', \eta'')\, p_m(0, \eta'', \eta) \frac{d\eta''}{\eta''} \quad (25)$$

and (13) implies for $p_m(0, \eta', \eta)$:

$$p_m(0, \eta', \eta) = \frac{\lambda}{4\pi}\left[(2 - \delta_{0,m}) \sum_{i=m}^{n} c_i^m P_i^m(\eta')P_i^m(\eta) + 2\eta \sum_{i=m}^{n} c_i^m P_i^m(\eta') \cdot \int_0^1 P_i^m(-\eta'')\rho_m(\eta'', \eta)d\eta''\right] \quad (26)$$

Quantities $\rho_m(\eta', \eta)$ included in (26) are determined from the expansion

$$\rho(\eta', \eta, \varphi' - \varphi) = \sum_{m=0}^{n} \rho_m(\eta', \eta) \cos m(\varphi' - \varphi) \quad (27)$$

Substitution of (24) and (27) in (23) gives

$$(\eta + \zeta)\rho_m(\eta, \zeta) = \pi p_m(0, -\zeta, \eta) + \pi(1 + \delta_{0,m})\zeta \times \int_0^1 \rho_m(\eta'', \zeta) \cdot p_m(0, \eta'', \eta)d\eta'' \quad (28)$$

Using (26) and introducing auxiliary functions $\varphi_i^m(\eta)$ determined by the following equations:

$$\varphi_i^m(\eta) = P_i^m(\eta) + \frac{2\eta}{2 - \delta_{0,m}} \int_0^1 P_i^m(-\eta'')\rho_m(\eta'', \eta)d\eta'' \quad (29)$$

we have

$$\rho_m(\eta, \zeta) = \frac{\lambda}{4}(2 - \delta_{0,m}) \sum_{i=m}^{n} (-1)^{i+m} c_i^m \frac{\varphi_i^m(\eta)\varphi_i^m(\zeta)}{\eta + \zeta} \quad (30)$$

The equations for the auxiliary functions $\varphi_i^m(\eta)$ can be found readily by substituting (30) in (29) and are written in the following terms:

$$\varphi_i^m(\eta) = P_i^m(\eta) + \frac{\lambda}{2}\eta \sum_{k=m}^{n} (-1)^{i+k} c_k^m \varphi_k^m(\eta) \times \int_0^1 \frac{\varphi_k^m(\eta'')}{\eta + \eta''} P_i^m(\eta'')d\eta$$
$$(i = m, m + 1, \ldots, n;\ 0 \leqslant m \leqslant n) \quad (31)$$

The results expressed in terms of (27–31) were first derived by Ambartsumian[4] in another way. The derivation of these formulas demonstrated herein, employing the concept of the probability of the emission of a quantum from the medium permits one to reveal the probabilistic meaning of the auxiliary functions $\varphi_i^m(\eta)$. Namely, using (26) and (29) we have

$$p_m(0, \eta', \eta) = \frac{\lambda}{4\pi}(2 - \delta_{0,m}) \sum_{i=m}^{n} c_i^m P_i^m(\eta')\varphi_i^m(\eta) \quad (32)$$

Hence, (24) and (32) imply that functions $\varphi_i^m(\eta)$ determine the probability of the emission of a quantum from the medium when it is absorbed in the surface layer.

4. Luminosity of the Medium for Different Sources of Radiation

If function $q(\tau, \eta', \eta, \varphi' - \varphi)$ exists for a given medium, then the intensity of the radiation $I(0, \eta, \varphi)$ emitted from the medium can be found readily for any arrangement of the sources. This intensity can be determined according to the following equation:

$$I(0, \eta, \varphi) = \int_0^\infty \int_{-1}^{+1} \int_0^{2\pi} B_0(\tau, \eta', \varphi') \times q(\tau, \eta', \eta, \varphi' - \varphi) \frac{d\tau}{\eta} d\eta' d\varphi' \quad (33)$$

where $B_0(\tau, \eta, \varphi)d\tau d\omega$ is the amount of radiant energy emitted by the radiational sources located at optical depth τ in elementary space with a cross section of 1 cm^2 and an optical thickness $d\tau$ at an angle arc $\cos\eta$ to the outer normal and azimuth φ within solid angle $d\omega$ per sec. Hence, if the medium is illuminated by a parallel radiation, then one can write

$$B_0(\tau, \eta, \varphi) = \frac{\lambda}{4} Sx(\gamma_2)e^{-(\tau/\zeta)} \quad (34)$$

where

$$\cos\gamma_2 = -\eta\zeta + \sqrt{(1 - \eta^2)(1 - \zeta^2)} \cos(\varphi - \varphi_0)$$

substituting (34) in (33) and taking (1) into account, we derive (16) used in the foregoing section.

In a series of instances it appears possible to obtain $I \times (0, \eta, \varphi)$ without finding functions $q(\tau, \eta', \eta, \varphi' - \varphi)$ beforehand. Let us now study the determination of the intensity of radiation emitted from the medium for certain radiational sources.

1) Let the power of the radiational sources located in the medium be determined as follows:

$$B_0(\tau, \eta, \varphi) = b(\eta, \varphi) \cdot e^{-s\tau} \quad (35)$$

Then the following formula is derived for the intensity of radiation emitted from the medium:

$$I(0, \eta, \varphi) = \frac{1}{\eta} \int_{-1}^{+1} \int_0^{2\pi} b(\eta', \varphi')\bar{q}(s, \eta', \eta, \varphi' - \varphi)d\eta'd\varphi' \quad (36)$$

which is obtained when (35) is substituted in (33). In this formula

$$\bar{q}(s, \eta', \eta, \varphi' - \varphi) = \int_0^\infty e^{-s\tau} q(\tau, \eta', \eta, \varphi' - \varphi)d\tau \quad (37)$$

Quantity $\bar{q}(s, \eta', \eta, \varphi' - \varphi)$ is determined by using (6). Multiplication of both parts of this equation by $e^{-s\tau}d\tau$ and integration over τ from 0 to ∞ gives

$$\bar{q}(s, \eta', \eta, \varphi' - \varphi) = \frac{\eta' q(0, \eta', \eta, \varphi' - \varphi) + \bar{p}(s, \eta', \eta, \varphi' - \varphi)}{1 + s\eta'} \quad (38)$$

where

$$\bar{p}(s, \eta', \eta, \varphi' - \varphi) = \int_0^\infty e^{-s\tau} p(\tau, \eta', \eta, \varphi' - \varphi)d\tau \quad (39)$$

Equations (4), (5), and (14) give

$$q(0, \eta', \eta, \varphi' - \varphi) = \begin{cases} \dfrac{1}{\pi} \rho(-\eta', \eta, \varphi' - \varphi) \cdot \eta & \text{for } \eta' < 0 \\ \delta(\eta' - \eta) \cdot \delta(\varphi' - \varphi) & \text{for } \eta' > 0 \end{cases} \quad (40)$$

Quantity $\bar{p}(s, \eta', \eta, \varphi' - \varphi)$ can be determined from (12).

This quantity will be derived in the second part of this paper. Another way to derive this quantity is to use the generalized coefficient of brightness $F(\eta, -\eta', \zeta, \varphi - \varphi_0)$ introduced by Sobolev[5] and determined as follows:

$$SF(\eta, -\eta', \zeta, \varphi - \varphi_0)\zeta = \int_0^\infty B(\tau, \eta', \zeta, \varphi - \varphi_0)e^{-(\tau/\eta)}\frac{d\tau}{\eta} \quad (41)$$

Equations (9), (39), and (41) give

$$\bar{p}(s, \eta', \eta, \varphi' - \varphi) = \frac{1}{\pi}\cdot\frac{\eta}{s}F\left(\frac{1}{s}, \eta', \eta, \varphi' - \varphi\right) \quad (42)$$

The methods of deriving $F(\eta'', \eta', \eta, \varphi' - \varphi)$ for $0 \leqslant \eta'' \leqslant 1$ are shown in Ref. 5. For other values of η'' the magnitude of the generalized coefficient of brightness can be calculated readily according to the formulas given in Ref. 5.

The intensity of radiation emitted from the medium can be determined, in addition to (36), in terms of another formula which is derived readily when (40) is substituted in (36). It is as follows:

$$I(0, \eta, \varphi) = \frac{1}{\eta}\int_{-1}^{+1}\int_0^{2\pi}\frac{b(\eta', \varphi')}{1 + s\eta'} \times \bar{p}(s, \eta', \eta, \varphi' - \varphi)d\eta'd\varphi' - \frac{1}{\pi}\int_0^1\int_0^{2\pi}\frac{b(-\eta', \varphi)}{1 - s\eta'} \times \rho(\eta', \eta, \varphi' - \varphi)\eta'd\eta'd\varphi' + \frac{b(\eta, \varphi)}{1 + s\eta} \quad (43)$$

2) Let us now study the case when the power of the radiational sources located in medium $B_0(\tau)$ is dependent only on the optical depth and is not dependent on the direction. Then (33) gives for the intensity of the radiation emitted from the medium

$$I(0, \eta) = \frac{4\pi}{\eta}\int_0^\infty B_0(\tau)Q(\tau, \eta)d\tau \quad (44)$$

where

$$Q(\tau, \eta) = \frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} q(\tau, \eta', \eta, \varphi' - \varphi)d\eta'd\varphi' \quad (45)$$

Quantity $Q(\tau, \eta)$ determines the probability for a quantum, emitted at depth τ, to be re-emitted from the medium at the angle arc $\cos\eta$ to the normal. The analogous quantity for an absorbed quantum is determined as follows:

$$P(\tau, \eta) = \frac{1}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} p(\tau, \eta', \eta, \varphi' - \varphi)d\eta'd\varphi' \quad (46)$$

When integrated over η' and φ', Eq. (1) gives:

$$P(\tau, \eta) = \lambda Q(\tau, \eta) \quad (47)$$

and it follows from (12):

$$\frac{\partial P(\tau, \eta)}{\partial\tau} = -\frac{1}{\eta}P(\tau, \eta) + 2\pi\int_0^1 P(\tau, \eta')p_0(0, \eta', \eta)\frac{d\eta'}{\eta'} \quad (48)$$

$I(0, \eta)$ can be found very easily if $B_0(\tau)$ is a polynomial. Let

$$B_0(\tau) = b_0 + b_1\tau + b_2\tau^2 + \ldots \quad (49)$$

Hence we have from (44) and (47):

$$I(0, \eta) = \frac{4\pi}{\lambda}[b_0A_0(\eta) + b_1A_1(\eta) + b_2A_2(\eta) + \ldots] \quad (50)$$

where

$$A_m(\eta) = \int_0^\infty P(\tau, \eta)\tau^m\frac{d\tau}{\eta} \quad (51)$$

Let us now establish the method of determining $A_m(\eta)$. Multiplication of (48) by τ^m and integration over τ from 0 to ∞, and with the consideration that $m \geqslant 1$, gives

$$-m\eta A_{m-1}(\eta) = -A_m(\eta) + 2\pi\int_0^1 A_m(\eta')p_0(0, \eta', \eta)d\eta' \quad (52)$$

If (32) is used, then

$$-m\eta A_{m-1}(\eta) = -A_m(\eta) + \frac{\lambda}{2}\sum_{i=0}^{n} x_i\varphi_i^0(\eta)\int_0^1 A_m(\eta')P_i(\eta')d\eta' \quad (53)$$

Equation (53) appears to be recurrent and to make it possible to determine $A_m(\eta)$ from the known quantity $A_{m-1}(\eta)$. In order to determine the value of the integrals included in (53), a system of equations, derived by multiplying (53) by $P_i(\eta)$ and integrating over η from 0 to 1, must be solved. In order to determine $A_0(\eta)$ it is necessary to integrate (48) over τ from 0 to ∞. Hence we have

$$-P(0, \eta) = -A_0(\eta) + \frac{\lambda}{2}\sum_{i=0}^{n} x_i\varphi_i^0(\eta)\cdot\int_0^1 A_0(\eta')P_i(\eta')d\eta' \quad (54)$$

The integrals included in (54) are determined by the method described in the foregoing, and quantity $P(0, \eta)$, as clearly implied by (32), is as follows:

$$P(0, \eta) = \frac{\lambda}{4\pi}\varphi_0^0(\eta) \quad (55)$$

It is pointed out for the spherical indicatrix of scattering under condition (49) that the problem of radiation emitted from a medium has already been solved by Sobolev[2] and Ueno.[6]

Let us now investigate case $B_0(\tau) = 1$ in detail. It is clear that (50), (54), and (55) can be used here assuming that $b_0 = 1$ and $b_1 = b_2 = \ldots = 0$. However, the problem is solved in a different manner. Substitution of $b(\eta, \varphi) = 1$ and $s = 0$ in (43) which corresponds to $B_0 = 1$, gives

$$I(0, \eta) = \frac{4\pi}{\eta}\bar{P}(0, \eta) - \frac{1}{\pi}\int_0^1\int_0^{2\pi}\rho(\eta', \eta, \varphi' - \varphi)\eta'd\eta'd\varphi' + 1 \quad (56)$$

However, Eqs. (44) and (47) give

$$I(0, \eta) = \frac{1}{\lambda}\cdot\frac{4\pi}{\eta}\bar{P}(0, \eta) \quad (57)$$

Substitution of (57) in (56) gives

$$I(0, \eta) = \frac{1 - \frac{1}{\pi}\int_0^1\int_0^{2\pi}\rho(\eta', \eta, \varphi' - \varphi)\eta'd\eta'd\varphi'}{1 - \lambda} \quad (58)$$

In addition, as is known already (see Ref. 2), when the semi-infinite medium is illuminated by parallel beams incident at an angle arc $\cos\eta$ to the normal, the albedo of this medium $A(\eta)$ can be found as follows:

$$A(\eta) = \frac{1}{\pi}\int_0^1\int_0^{2\pi}\rho(\eta', \eta, \varphi' - \varphi)\eta'd\eta'd\varphi' \quad (59)$$

Comparison of (58) and (59) gives finally

$$I(0, \eta) = \frac{1 - A(\eta)}{1 - \lambda} \quad (60)$$

It should be noted that (60) is valid for any scattering indicatrix. If the indicatrix is spherical, then

$$A(\eta) = 1 - \varphi(\eta)\cdot\sqrt{1 - \lambda} \quad (61)$$

and from (60) the well-known formula is derived (see Ref. 2):

$$I(0, \eta) = \frac{\varphi(\eta)}{\sqrt{1 - \lambda}} \quad (62)$$

Now let the scattering indicatrix have the form

$$x(\gamma) = 1 + x_1 \cos\gamma \quad (63)$$

For such a case

$$A(\eta) = 1 - \frac{1}{\eta} \varphi_1^0(\eta) \quad (64)$$

and from (60) it follows that

$$I(0, \eta) = \frac{1}{\eta} \cdot \frac{\varphi_1^0(\eta)}{1 - \lambda} \quad (65)$$

3) Let us now determine the angular distribution of the intensity of radiation $I(\eta)$ diffusely transmitted by the medium which has a large optical depth. In such a case, it should be considered that the radiational sources are located at an infinitely large depth.

Let us first derive the equations for the emission probability of a quantum from a very large optical depth ($\tau \gg 1$). Physical meaning implies that this probability will be dependent only on τ and η and, in addition, that the expression for $P(\tau, \eta)$ must be sought in the following form:

$$P(\tau, \eta) = C(\eta) \cdot e^{-k\tau} \quad (66)$$

where k is an unknown constant and $C(\eta)$ is an unknown function.

Substitution of (66) in (48) gives

$$-kC(\eta) = -\frac{1}{\eta} C(\eta) + 2\pi \int_0^1 C(\eta')p_0(0, \eta', \eta) \frac{d\eta'}{\eta'} \quad (67)$$

Equations (67) and (32) indicate that

$$C(\eta) = \frac{\lambda}{2} \eta \sum_{i=0}^{n} \frac{c_i x_i \varphi_i^0(\eta)}{1 - k\eta} \quad (68)$$

where

$$c_i = \int_0^1 C(\eta)P_i(\eta) \frac{d\eta}{\eta} \quad (69)$$

Equation (68) determines function $C(\eta)$. Constants c_i included in (68) are derived from the system of linear algebraic equations

$$c_j = \frac{\lambda}{2} \sum_{i=0}^{n} x_i c_i a_{ij} \quad (70)$$

where

$$a_{ij} = \int_0^1 \frac{\varphi_i^0(\eta)P_j(\eta)}{1 - k\eta} d\eta \quad (71)$$

System (70) is derived by multiplying (68) by $P_j(\eta) \cdot (1/\eta)$ and integrating over η from 0 to 1. The condition of nontrivial solution of this system determines the value of constant k. It should be pointed out that this method determines $C(\eta)$ only to within a constant factor.

Taking (66), (68), (33), and (47) into account for the angular distribution of the intensity of diffusely transmitted radiation we have

$$I(\eta) = \frac{\lambda}{2} \sum_{i=0}^{n} \frac{x_i c_i \varphi_i^0(\eta)}{1 - k\eta} \quad (72)$$

The investigated problem was previously solved by Ambartsumian[7] by the method of combined layers which he developed. Solution by the probabilistic method described here leads to the goal much faster and therefore may be of some interest.

The present paper solves the problem of determining the intensity of radiation emitted from a semi-infinite medium with an arbitrary scattering indicatrix for any distribution of sources, the power of which, in a general case, can be dependent on depth and direction. The solution is obtained by using the probabilistic concept of emission of a quantum from the medium for a given direction.

It is demonstrated by Sobolev[2, 3] that on the basis of the emission probability of a quantum from the medium it is possible to determine also the radiation field within the medium. The luminous mechanism within the medium will be determined in Part II. Application of the Laplace transforms to the problems under investigation will be demonstrated there also.

Summary

The problem of diffusion of radiation in a semi-infinite medium with nonspherical indicatrix of scattering is considered with the aid of the probabilistic method. In Sec. 1 the probability of emergence for a photon from a medium in a given direction is introduced. In Sec. 2 the equations for the probability of emergence for a photon from a medium are given. In Sec. 3 the intensity of radiation emerging from a medium illuminated by parallel rays is derived. The expressions for the intensity of radiation emerging from a medium for different sources of radiation are derived in Sec. 4.

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Reviewer's Comment

In this paper, the author extends Sobolev's method of the solution of radiative transfer problem in semi-infinite medium for anisotropic scattering. This solution is based on the probabilistic method, whereas Chandrasekhar[1] indicated the solution of the same problem based on the use of the equation of radiative transfer and the principles of invariance. The equivalence of these two different methods has been demonstrated for the case of isotropic scattering by Ueno.[2] Chandrasekhar's method is also discussed in an exact mathematical analysis by Busbridge.[3]

The problem, investigated by the author in the present paper, can be considered a special case of the problem of radiative transfer in a plane-parallel medium of finite optical depth, studied for anisotropic scattering by Churchill et al.[4] and by Yanovitski.[5] The latter author indicated the solution of an inhomogeneous medium, in which the albedo for single scattering, as well as the indicatrix of scattering (phase function), vary with optical depth. Yanovitski uses the auxiliary equation (for the source function) as the starting point for his discussion, whereas Churchill et al. extend Chandrasekhar's solution for the case of variable albedo for single scattering. By a proper limiting process, in which the